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Solution of the master equation for the Bak-Sneppen model of biological evolution in a finite ecosystem

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The master equations describing processes of biological evolution in the framework of the random neighbor Bak-Sneppen model are studied. For the ecosystem of N species they are solved exactly and asymptotical behavior of this solution for large N is analyzed. [S1063-651X(97)50608-8]

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INTRODUCTION

The model of biological evolution proposed by Bak and Sneppen [1,2] describes mutation and natural selection of interacting species. It is the dynamical system that is defined as follows. The state of the ecosystem of N species is characterized by a set $\{x_1,\ldots,x_N\}$ of N number, $1\geqslant x_i\geqslant 0$. In so doing, x_i represents the barrier toward further evolution of the species. Initially, each x_i is set to a randomly chosen value. At each time step the barrier x_i with minimal value and $K-1$ other barriers are replaced by K new random numbers. In the random neighbor model (RNM), which will be considered in this paper the $K-1$ replaced nonminimal barriers are chosen at random.

The RNM is the simplest model describing the avalanchelike processes, which are supposed by a conception of ''punctuated equilibrium'' in biological evolution. These processes are the most characterizing features for self-organized criticality recently intensively investigated both numerically and analytically [3–6]. The RNM is more convenient for analytical studies. The master equations obtained in [3] for RNM are very useful for this aim. In [7] the explicit solution of master equations was found for an infinite ecosystem. In this paper we solve the master equations for a finite number N of species in an ecosystem. We restrict ourselves to the simplest case $K=2$.

MASTER EQUATIONS FOR RNM

The master equations for the RNM are obtained in [3]. They are of the form

$$P_n(t+1)=A_nP_n(t)+B_{n+1}P_{n+1}(t)+C_{n-1}P_{n-1}(t)+D_{n+2}P_{n+2}(t)+(B_1\delta_{n,0}+A_1\delta_{n,1}+C_1\delta_{n,2})P_0(t). \quad (1)$$

Here, $P_n(t)$ is the probability that n is the number of barriers having values less than a fixed value λ at the time t; $0\leqslant n\leqslant N$, $0\leqslant\lambda\leqslant 1$, $0\leqslant t$; $P_n(0)$ are proposed to be given. For $0<n\leqslant N$

$$A_n=2\lambda(1-\lambda)+\frac{n-1}{N-1}\lambda(3\lambda-2),$$

$$B_n=(1-\lambda)^2+\frac{n-1}{N-1}(1-\lambda)(3\lambda-1),$$

$$C_n=\lambda^2-\frac{n-1}{N-1}\lambda^2,\quad D_n=(1-\lambda)^2\frac{n-1}{N-1}, \quad (2)$$

and $A_n=B_n=C_n=D_n=0$ for $n=0,n>N$.

By virtue of the definition of $P_n(t)$,

$$P_n(t)\geqslant 0, \quad (3)$$

$$\sum_{n=0}^{N}P_n(t)=1. \quad (4)$$

Summing in Eq. (1) over n and taking into account Eq. (2) it is easy to establish that

$$\sum_{n=0}^{N}P_n(t+1)=\sum_{n=0}^{N}P_n(t). \quad (5)$$

Therefore, if $P_n(0)$ are chosen in such a way that Eq. (4) is fulfilled for $t=0$, then by virtue of Eq. (5) it is the case for the solution of Eq. (1) for $t>0$ too. For analysis of Eq. (1) it is convenient to introduce the generating function $q(z,u)$:

$$q(z,u)\equiv\sum_{t=0}^{\infty}\sum_{n=0}^{N}P_n(t)z^nu^t. \quad (6)$$

By virtue of Eqs. (3) and (4) $q(z,t)$ is a polynomial in z, analytical in u for $|u|<1$, and

$$q(1,u)=\frac{1}{1-u}. \quad (7)$$

The master equations (1) can be rewritten for the generating function $q(z,u)$ as follows:

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$$\frac{1}{u}[q(z,u)-q(z,0)]=(1-\lambda+\lambda z)^2\left\{\frac{1}{z}\left[1-\frac{1-z}{N-1}\left(\frac{1}{z}-\frac{\partial}{\partial z}\right)\right]\times[q(z,u)-q(0,u)]+q(0,u)\right\}. \quad (8)$$

The function $q(z,0)=\Sigma_{n=0}^{N}P_n(0)z^n$ in Eq. (8) is assumed to be given.

ASYMPTOTIC EXPANSION OF $q(z,u)$ FOR LARGE N

If the function $q(z,0)$ has an asymptotic expansion in the region of large N of the form

$$q(z,0)=\sum_{k=0}^{\infty}\frac{q_k(z,0)}{(N-1)^k}$$

then Eq. (8) enables one to obtain the similar asymptotic expansion for $q(z,u)$,

$$q(z,u)=\sum_{k=0}^{\infty}\frac{q_k(z,u)}{(N-1)^k}.$$

The main approximation of $q(z,u)$, the function $q_0(z,u)$, can be found from the equation

$$[z-u(1-\lambda+\lambda z)^2]q_0(z,u)=zq_0(z,0)+u(1-\lambda+\lambda z)^2(z-1)q_0(0,u) \quad (9)$$

following from Eq. (8). Since $q_0(z,u)$ is analytical for $|z|<1$, $|u|<1$,

$$0=\alpha q_0(\alpha,0)+u(1-\lambda+\lambda\alpha)^2(\alpha-1)q_0(0,u), \quad (10)$$

where

$$\alpha=\alpha(u)=\frac{1-2\lambda(1-\lambda)u-[1-4\lambda(1-\lambda)u]^{1/2}}{2\lambda^2u} \quad (11)$$

is the solution of $\alpha-u(1-\lambda+\lambda\alpha)^2=0$. Obviously, $|\alpha|<1$ for sufficiently small $|u|$. Thus, from Eq. (10) the function $q_0(0,u)$ can be found,

$$q_0(0,u)=\frac{q_0(\alpha,0)}{1-\alpha}. \quad (12)$$

Substituting Eq. (12) in the right-hand side of Eq. (9), one can find its solution in the following form:

$$q_0(z,u)=\frac{zq_0(z,0)(1-\alpha)+(z-1)u(1-\lambda+\lambda z)^2q_0(\alpha,0)}{[z-u(1-\lambda+\lambda z)^2](1-\alpha)}, \quad (13)$$

where $\alpha(u)$ is defined by (11).

For $k>0$ the functions $q_k(z,u)$ are defined by recurrent relations

$$q_k(z,u)=\frac{u(1-\lambda+\lambda z)^2(z-1)q_k(\alpha,0)+(1-\alpha)zq_k(z,0)}{[z-u(1-\lambda+\lambda z)^2](1-\alpha)}+\frac{(1-z)u(1-\lambda+\lambda z)^2[r_{k-1}(z,u)-r_{k-1}(\alpha,u)]}{z-u(1-\lambda+\lambda z)^2}. \quad (14)$$

Here,

$$r_k(z,u)\equiv z\frac{\partial}{\partial z}\frac{q_k(z,u)-q_k(0,u)}{z}. \quad (15)$$

By virtue of Eqs. (14) and (15) the first correction to the lowest approximation (13) of $q(z,u)$ has the form

$$q_1(z,u)=\frac{u(1-\lambda+\lambda z)^2(z-1)q_1(\alpha,0)+(1-\alpha)zq_1(z,0)}{[z-u(1-\lambda+\lambda z)^2](1-\alpha)}+\frac{(1-z)u(1-\lambda+\lambda z)^2[r_0(z,u)-r_0(\alpha,u)]}{z-u(1-\lambda+\lambda z)^2} \quad (16)$$

where

$$r_0(z,u)=z\frac{\partial}{\partial z}\frac{(1-\alpha)q_0(z,0)+[u(1-\lambda+\lambda z)^2-1]q_0(\alpha,0)}{[z-u(1-\lambda+\lambda z)^2](1-\alpha)}. \quad (17)$$

EXACT FORM OF $q(z,u)$

Let us introduce the quantity

$$Q(z,u)\equiv\frac{q(z,u)-q(0,u)}{z}. \quad (18)$$

It follows from Eq. (8) that this function fulfills the relation of the form

$$\left(z-u(1-\lambda+\lambda z)^2-\frac{u(1-\lambda+\lambda z)^2(1-z)}{N-1}\frac{\partial}{\partial z}\right)Q(z,u)=q(z,0)+[u(1-\lambda+\lambda z)^2-1]q(0,u). \quad (19)$$

This inhomogeneous differential equation for $Q(z,u)$ has a special solution

$$Q(z,u)=(N-1)e^{R(z,u)}\int_z^1 e^{-R(x,u)}g(x,u)dx\equiv Q_{sp}(z,u). \quad (20)$$

Here,

$$R(z,u)=\frac{N-1}{u}\left(\ln(1-\lambda+\lambda z)-(1-u)\ln(1-z)+\frac{1-\lambda}{\lambda(1-\lambda+\lambda z)}\right), \quad (21)$$

$$g(x,u)=\frac{q(x,0)+[u(1-\lambda+\lambda x)^2-1]q(0,u)}{u(1-\lambda+\lambda x)^2(1-x)} \quad (22)$$

and the derivative of $R(z,u)$ with respect to z has the form

$$\frac{\partial R(z,u)}{\partial z}=\frac{(N-1)[z-u(1-\lambda+\lambda z)^2]}{u(1-\lambda+\lambda z)^2(1-z)}. \quad (23)$$

General solution of the corresponding to Eq. (19) homogeneous equation

$$\left(z-u(1-\lambda+\lambda z)^2-\frac{u(1-\lambda+\lambda z)^2(1-z)}{N-1}\frac{\partial}{\partial z}\right)S(z,u)=0 \quad (24)$$

is of the form

$$S(z,u)=F(u)e^{R(z,u)}. \quad (25)$$

Here, $F(u)$ is an arbitrary function of u. Hence, it follows from Eqs. (19), (20), and (25) that the function $Q(z,t)$ can be represented as follows:

$$Q(z,u)=F(u)e^{R(z,u)}+Q_{sp}(z,u). \quad (26)$$

By virtue of initial condition (7) for $q(z,u)$,

$$Q(1,u)=\frac{1}{1-u}-q(0,u). \quad (27)$$

For $0<u<1, z\to 1, S(z,u)$ diverges and $S_{sp}(z,u)$ has the finite limit

$$\lim_{z\to 1}Q_{sp}(z,u)=\frac{1}{1-u}-q(0,u). \quad (28)$$

Hence, $F(u)=0$ in Eq. (26) and this representation for $Q(z,u)$ can be rewritten in the form

$$Q(z,u)=(N-1)e^{R(z,u)}\int_{(\lambda-1)/\lambda}^{1}e^{-R(x,u)}g(x,u)dx+(N-1)e^{R(z,u)}\int_{z}^{(\lambda-1)/\lambda}e^{-R(x,u)}g(x,u)dx. \quad (29)$$

It follows from Eq. (18) that

$$Q\left(\frac{\lambda-1}{\lambda},u\right)=\frac{\lambda(q[(\lambda-1)/\lambda,u]-q(0,u))}{\lambda-1}.$$

For the terms in the right-hand side of Eq. (29) we have for $0<u<1, 0<\lambda<1$

$$\lim_{z\to(\lambda-1)/\lambda+0} e^{R(z,u)}=+\infty, \quad (30)$$

$$\lim_{z\to(\lambda-1)/\lambda+0}(N-1)e^{R(z,u)}\int_{z}^{(\lambda-1)/\lambda}e^{-R(x,u)}g(x,u)dx=\frac{\lambda(q[(\lambda-1)/\lambda,u]-q(0,u))}{\lambda-1}. \quad (31)$$

Therefore Eq. (29) can represent the function $Q(z,u)$ with necessary analytical properties only if

$$\int_{(\lambda-1)/\lambda}^{1}e^{-R(x,u)}g(x,u)dx=0. \quad (32)$$

This equation defines the function $q(0,u)$,

$$q(0,u)=\frac{\int_{(\lambda-1)/\lambda}^{1}e^{-R(x,u)q(x,0)[(1-\lambda+\lambda x)^2(1-x)]^{-1}dx}}{\int_{(\lambda-1)/\lambda}^{1}e^{-R(x,u)[1-u(1-\lambda+\lambda x)^2][(1-\lambda+\lambda x)^2(1-x)]^{-1}dx}} \quad (33)$$

Thus, we obtain from Eqs. (29) and (32) the solution of Eq. (8) in the following form:

$$q(z,u)=z\,\frac{N-1}{u}\,e^{R(z,u)}\int_z^{(\lambda-1)/\lambda}e^{-R(x,u)}\frac{q(x,0)dx}{(1-\lambda+\lambda x)^2(1-x)}+q(0,u)\left(1-z\,\frac{N-1}{u}\,e^{R(z,u)}\int_z^{(\lambda-1)/\lambda}e^{-R(x,u)}\frac{[1-u(1-\lambda+\lambda x)^2]dx}{(1-\lambda+\lambda x)^2(1-x)}\right), \quad (34)$$

where $q(0,u)$ is defined by Eq. (33).

CONCLUSION

We constructed the solution of the master equation (8) for the finite number N of species in the ecosystem. It can be proven that the main term (13) of its asymptotic for large N coincides with the one obtained in [7]. Using Eq. (34) one can obtain all the known analytical results for RNM. One can hope that it helps to understand better the most important properties of the self-organized criticality processes.

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